40. $S_n = \sum_{j=1}^{n} X_j$

$\frac{S_n}{n} \xrightarrow{a.s.} 0 \iff$ 1. $\frac{S_n}{n} \xrightarrow{P} 0$

2. $\frac{S_{2^n}}{2^n} \xrightarrow{a.s.} 0$

$(\Rightarrow)$

1. a.s $\Rightarrow$ P

2. Subsequence.

$(\Leftarrow)$ 1. $\not\Rightarrow \frac{S_n}{n} \xrightarrow{a.s.} 0$

$\sum X_j$ converges a.s. $\iff \sum X_j$ converges in P

(X_j - independent).

$\frac{S_n}{n} \xrightarrow{P} 0$ $\quad \forall \varepsilon > 0 \quad P\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) \to 0$ as $n \to \infty$

$\frac{S_{2^n}}{2^n} \xrightarrow{a.s} 0 \qquad n_k \quad \left|\frac{n_{k+1}}{n_k}\right| < C.$

$2^k \le n < 2^{k+1}$

$\frac{S_n}{n} \le 2 \frac{S_n}{2^k}$

$\left\{\frac{|S_n|}{n} > \varepsilon\right\} = \{|S_n| > n\varepsilon\} \subset \{|S_n| > 2^k \varepsilon\}$

? $P\left\{\max_{n \in [2^k, 2^{k+1}]} |S_n| > 2^k \varepsilon\right\} \to 0$?

$P\left(\left\{\max_{n \le 2^{k+1}} |S_n| > 2^k \varepsilon\right\}\right) \le \frac{1}{1-c} P\left(|S_{2^{k+1}}| > 2^k \varepsilon\right)$

$\frac{S_{2^{k+1}}}{2^{k+1}} > \frac{\varepsilon}{2} \quad k \to \infty$

$c = \max_{n \le 2^{k+1}} \left(P\{|S_{2^{k+1}} - S_n| > 2^k \varepsilon\}\right) \overset{??}{<} 1$

$P\left(\{|S_{2^{k+1}} - S_n| > 2^k \varepsilon\}\right) \le$

$P\left(|S_{2^{k+1}}| > 2^{k-1} \varepsilon\right) + P\left(|S_n| > 2^{k-1} \varepsilon\right)$

$$P\left(|S_{2^{k+1}}| > 2^{k-1}\varepsilon\right) + P\left(|S_n| > 2^{k-1}\varepsilon\right) \to 0$$

$$\frac{S_n}{n} \to 0$$

So, for $n > N$, $P\left(\frac{|S_n|}{n} > \frac{\varepsilon}{4}\right) < \delta$

$\forall \delta > 0$

$2^{k+1} \geq n > N$

$$\{|S_n| > 2^{k-1}\varepsilon\} \subset \{|S_n| > n\tfrac{\varepsilon}{4}\}$$

$n \leq 2^{k+1}$

$$P(|S_n| > 2^{k-1}\varepsilon) < \delta.$$

$n \leq N.$

$$\{|S_n| > 2^{k-1}\varepsilon\} = \left\{\frac{|S_n|}{n} > \frac{2^{k-1}}{n}\varepsilon\right\}$$

$$\subset \left\{\frac{|S_n|}{n} > \frac{2^{k-1}}{N}\varepsilon\right\}$$

$\frac{|S_n|}{n}$ is finite a.s

So $\lim_{k\to\infty} \max_{n \leq N} P\left(\frac{|S_n|}{n} > \frac{2^{k-1}}{N}\varepsilon\right) = 0.$

(4) $X_1 + X_2 + \dots + X_n \sim N(\mu_1 + \mu_2 + \dots + \mu_n, \sigma_1^2 + \dots + \sigma_n^2)$

$\sigma_1^2 + \dots + \sigma_n^2 \to \infty$

$P(|X_1 + \dots + X_n| > c) \to 1$ $\forall \varepsilon$

So $\sum \sigma_j^2 < \infty$

By Kolmogorov, $\sum (X_j - \mu_j)$ a.s $\Rightarrow$

$\sum \mu_j$ - converges

Bonus Create μ_j, σ_j

$\sum \mu_j$ - diverges.

$\sum \sigma_j^2 = \infty$

X — distribution, $E(X)=0$, $E(X^2)=1$

$X_i \overset{d}{=} \mu_i + \sigma_i X$, X_i — independent.

$\sum X_i$ — converges.

6) $e^{-\sum X_k} > 0$ a.s.

$$0 < E\left(e^{-\sum X_k}\right) = \prod \underline{E\left(e^{-X_k}\right)}$$

28) $E(X_n)=\mu$, $Var(X_n) \le \rho(0)$, $\underline{Cov(X_i, X_j) = 0}$, $i \ne j$

$$E\left(\left((X_1-\mu)+\dots+(X_n-\mu)\right)^2\right) \le$$

$$n\rho(0)$$

$$E\left(\left(\frac{(X_1-\mu)+\dots+(X_n-\mu)}{n}\right)^2\right) \le \frac{\rho(0)}{n} \to 0$$

$\to$ in L^2 0

$E(X_n)=\mu$, $Var(X_n) \le \rho(0)$

$Cov(X_i, X_j) \le \rho(i-j)$.

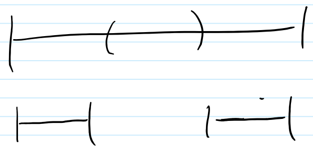
$$E\left(\left(\frac{(X_1-\mu)+\dots+(X_n-\mu)}{n}\right)^2\right) \le$$

$$\frac{1}{n^2}\left(n\rho(0) + 2n\rho(1) + 2n\rho(2) + \dots + 2n\rho(n-1)\right)$$

$$\le 2\,\frac{\rho(0)+\dots+\rho(n-1)}{n} \to 0$$

$\rho(n) \to 0$ Tauber.

38).



$y = 0.y_1 \dots y_n \dots$

H H H H $y_n \in \{0, 1, 2\}$. base 3

$y: y_{ij} \neq 1$.

$$P(A) = P\left(\sum X_{ij}/3^j \in A\right).$$

3) $E\left(\frac{N_n^2}{n^2}\right) \xrightarrow{?} 0.$